Chapter 2: Fourier Series

1.0 Introduction

• Fourier Series: representation of periodic signals as weighted sums of harmonically related frequencies.

• If a signal $x(t)$ is periodic signal, then $x(t)$ can be represented in terms of Fourier Series either in:
  a) Trigonometric Form
  b) Exponential (Complex) Form

Chapter 3: Fourier Series

1.1 Trigonometric Fourier Series

• General equation:

$$ x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] $$

• $x(t)$ is expressed as the sum of sinusoidal components having different frequencies where:

* $a_n$ and $b_n$: the Fourier coefficients
* $a_0$: DC Value

$$ a_n = \frac{2}{T} \int_{0}^{T} x(t) \cos n\omega_0 t \, dt $$
$$ b_n = \frac{2}{T} \int_{0}^{T} x(t) \sin n\omega_0 t \, dt $$

F0urier

Saw tooth waveform

Infinite number of sine waves (harmonics)
Example
Express the following signal $x(t)$ as shown in the figure below as Trigonometric Fourier Series:

$$x(t) = \begin{cases} \frac{t}{\pi} & ; 0 < t < \pi \\ 0 & ; \pi < t < 2\pi \end{cases}$$
Example
Express the following signal $x(t)$ as shown in figure below as Trigonometric Fourier Series:

1.2 Symmetry Properties

- The symmetry properties can be classified into 5 types :
  a) Even Symmetry
  b) Odd Symmetry
  c) Half Wave Symmetry
  d) Even And Half Wave Symmetry
  e) Odd And Half Wave Symmetry
1.2 Symmetry Properties

a) Even Symmetry

The signal $x(t)$ is said to be even symmetry if $x(t) = x(-t)$

b) Odd Symmetry

The signal $x(t)$ is said to be odd symmetry if $x(t) = -x(-t)$
1.2 Symmetry Properties

c) Half Wave Symmetry

The signal $x(t)$ is said to be half wave symmetry if $x(t) = -x(t + T/2)$

Example

The first half-cycle of a periodic signal is shown in figure below and the period is $T = 2$ sec. Sketch $y(t)$ clearly for 3 complete cycles if:

i) $y(t)$ is an even-symmetrical signal

ii) $y(t)$ is an odd-symmetrical signal

iii) $y(t)$ is a half-wave symmetrical signal
Chapter 3 : Fourier Series

Answer

i) even-symmetrical signal 

iii) odd-symmetrical signal

\[ x(t) \]

\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \]

\[ t \]

\[ x(t+1) \]

\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \]

\[ t \]

Chapter 3 : Fourier Series

Answer

iii) half-wave symmetrical signal

\[ x(t+1) \]

\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \]

\[ t \]
Example
The first half-cycle of a periodic signal is shown in figure below and the period is $T = 4$ sec. Sketch $y(t)$ clearly for 3 complete cycles if:

i) $y(t)$ is an even-symmetrical signal

ii) $y(t)$ is an odd-symmetrical signal

iii) $y(t)$ is a half-wave symmetrical signal

Answer

i) even-symmetrical signal

\[ x(t) \]

iii) odd-symmetrical signal

\[ -x(-t) \]
iii) half-wave symmetrical signal

1.2 Symmetry Properties

d) Even And Half-Wave Symmetry

Consider a half cycle signal $x(t)$ shown in figure below where $T = 4$ sec

For one complete cycle, the shape of signal $x(t)$ is the same properties.
1.2 **Symmetry Properties**

d) **Even And Half-Wave Symmetry**

Thus, the half-wave even symmetry signal $x(t)$ for 3 complete cycles is shown below:

![Half-wave even symmetry signal](image)


e) **Odd And Half-Wave Symmetry**

Consider a half cycle signal $x(t)$ shown in figure below where $T = 4$ sec

![Half cycle signal](image)

For one complete cycle, the shape of signal $x(t)$ is the same properties.
1.2 **Symmetry Properties**

e) **Odd And Half-Wave Symmetry**

Thus, the half-wave even symmetry signal \( x(t) \) for 3 complete cycles is shown below:

![Graph showing half-wave odd symmetry](Image)

\[ x(t) = \text{half-wave odd} \]

1.3 **Effects of Symmetry Properties**

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>( a_n )</th>
<th>( b_n )</th>
<th>( x(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Even</strong></td>
<td>( \frac{1}{T} ) [Area for 1T]</td>
<td>( \frac{2}{T} \int_{0}^{T/2} x(t) \cos n \omega_0 t , dt )</td>
<td>0</td>
</tr>
<tr>
<td><strong>Odd</strong></td>
<td>0</td>
<td>0</td>
<td>( \frac{2}{T} \int_{0}^{T/2} x(t) \sin n \omega_0 t , dt )</td>
</tr>
<tr>
<td><strong>Half-Wave</strong></td>
<td>0</td>
<td>( \frac{2}{T} \int_{0}^{T/2} x(t) \cos n \omega_0 t , dt ; n = \text{even} )</td>
<td>( \frac{2}{T} \int_{0}^{T/2} x(t) \sin n \omega_0 t , dt ; n = \text{odd} )</td>
</tr>
</tbody>
</table>
1.3 Effects of Symmetry Properties

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>$a_0$</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$x(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even and Half-Wave</td>
<td>0</td>
<td>$\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_0 t , dt$; $n = \text{even}$</td>
<td>0</td>
<td>$\sum_{n=0}^{\infty} a_n \cos n\omega_0 t$</td>
</tr>
<tr>
<td>Odd and Half-Wave</td>
<td>0</td>
<td>0</td>
<td>$\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t , dt$; $n = \text{odd}$</td>
<td>$\sum_{n=0}^{\infty} b_n \sin n\omega_0 t$</td>
</tr>
</tbody>
</table>

Hidden

To find the point of axis to yield the symmetry properties:

$x(t) = [a_i / \text{changes of } y \text{- axis}] + [\text{Fourier series of new } x(t) \text{ which taken as } g(t)]$
Example
Express the following signal $x(t)$ as shown in figure below as Trigonometric Fourier Series using symmetry property.

![Signal Diagram]

1.4 Hidden Symmetry

Example
Express signal $x(t)$ as TFS

![Signal Diagram]

Solution

$x(t)$ does not possess any symmetry properties.
1.4 Hidden Symmetry

Solution
However applying the hidden symmetry by shifting 0.5A of the DC value for signal x(t) will result signal below:

Now, signal g(t) possesses odd symmetry, thus FS of x(t):

\[ x(t) = 0.5A + \text{FS of signal } g(t) \]

---

Example
Express signal x(t) as TFS
1.4 Hidden Symmetry

Solution

Signal $x(t)$ posses even-symmetry property. Thus FS of $x(t)$ is:

$$x(t) = \pi/2 + \text{FS of signal } g(t)$$

1.5 Trigonometric Fourier Series

The trigonometric FS of signal $x(t)$ is given as:

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos n \omega \cdot t + b_n \sin n \omega \cdot t \right]$$

Euler’s Identity

$$\cos \theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \sin \theta = \frac{1}{2i}(e^{j\theta} - e^{-j\theta})$$

The expression $(a_n \cos n \omega \cdot t + b_n \sin n \omega \cdot t)$ can be expressed as follows:

$$a_n \cos n \omega \cdot t + b_n \sin n \omega \cdot t = \frac{a_n}{2} (e^{jn\omega \cdot t} + e^{-jn\omega \cdot t}) + \frac{b_n}{2i} (e^{jn\omega \cdot t} - e^{-jn\omega \cdot t})$$

$$= \frac{1}{2} (a_n - jb_n) e^{jn\omega \cdot t} + \frac{1}{2} (a_n + jb_n) e^{-jn\omega \cdot t}$$
1.5 Complex Fourier Series

Thus \( x(t) \):

\[
x(t) = a_0 + \sum_{n=1}^{\infty} \left[ \frac{1}{2}(a_n - jb_n) e^{i\omega_0 n t} + \frac{1}{2}(a_n + jb_n) e^{-i\omega_0 n t} \right]
\]

Let:

\[
g_n = a_n - jb_n \quad \quad \quad \quad \quad c_n = \frac{1}{2}(a_n + jb_n)
\]

Then \( x(t) \):

\[
x(t) = c_0 + \sum_{n=1}^{\infty} \left[ g_n e^{i\omega_0 n t} + c_n e^{-i\omega_0 n t} \right]
\]
1.5 **Complex Fourier Series**

Where:

\[
C_0 = a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, dt
\]

\[
C_n = \frac{1}{T} \left[ a_n - j b_n \right]
\]

\[
= \frac{1}{T} \left[ \frac{2}{T} \int_{0}^{T/2} x(t) \cos \omega_n t \, dt + \frac{2}{T} \int_{0}^{T/2} x(t) \sin \omega_n t \, dt \right]
\]

\[
= \frac{1}{T} \int_{0}^{T} x(t) \left[ \cos \omega_n t + j \sin \omega_n t \right] \, dt
\]

\[
C_n = \frac{1}{T} \int_{0}^{T} x(t) e^{-j \omega_n t} \, dt
\]

where: \( e^{j \theta} = \cos \theta + j \sin \theta \) (Euler’s Identity)

---

**Example**

Express the following signal \( x(t) \) as shown in figure below as an Exponential Fourier Series.

[Graph of a periodic signal with positive and negative values between -T and T]

\[ x(t) \]

\[ t \]

\[ T \]

\[ -0.5T \]

\[ T \]

\[ -1 \]

\[ 1 \]
Example
Express the following signal $x(t)$ as shown in figure below as an Exponential Fourier Series.

Example
The following figure below shows a half cycle of a periodic signal $x(t)$. If $x(t)$ is an odd symmetry signal, determine its EFS:
**Chapter 2: Fourier Series**

**Solution**

$x(t)$ is an odd symmetry signal

\[ T = 4 \]
\[ \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2} \]

Odd symmetry

\[ a_0 = 0 \ (C_0) \]
\[ a_n = 0 \]
\[ b_n = ??? \]

---

**1.6 Frequency Spectrum**

- Frequency spectrum consist of:
  
  a) **Amplitude spectrum**
  
  the plot of $|C_n|$ against $n\omega_0$

  ![Amplitude Spectrum Diagram]

  b) **Phase spectrum**

  the plot of $\angle C_n$ against $n\omega_0$

  ![Phase Spectrum Diagram]
1.6 Frequency Spectrum

- For complex number $C_n = a + jb$

  \[ \text{Magnitude, } |C_n| = \sqrt{a^2 + b^2} \]

  \[ \text{Phase, } \angle C_n = \frac{\tan^{-1}\frac{b}{a}}{\pi} \]

- Conditions
  
  i) If $b = 0$, $C_n = a$
  
  \[ |C_n| = a \]
  
  \[ \angle C_n = 0 \text{ if } a > 0 \text{ (positive)} \]
  
  \[ = \pi \text{ or } -\pi \text{ if } a < 0 \text{ (negative)} \]

  ii) If $a = 0$, $C_n = jb$
  
  \[ |C_n| = b \]
  
  \[ \angle C_n = \pm \pi/2 \text{ if } b > 0 \text{ (positive)} \]
  
  \[ = -\pi/2 \text{ if } b < 0 \text{ (negative)} \]
Example
Express the following signal $x(t)$ as shown in figure below as an Exponential Fourier Series. Plot the frequency spectrum of signal $x(t)$.

![Diagram](image)

---

Example
Express the following signal $x(t)$ as shown in figure below as an Exponential Fourier Series. Plot the frequency spectrum of signal $x(t)$.

![Diagram](image)
Example
The Fourier Series of signal $x(t)$ is given as:

$$x(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin nt$$

Sketch the frequency spectrum of $x(t)$ for $n = \pm 5, \pm 4, \pm 3, \pm 2, \pm 1$ and 0

1.7 TFS Coefficients and EFS Coefficients Relationship

- $c_0 = a_0$
- $c_n = \frac{1}{2}(a_n - jb_n)$ or $c_{-n} = \frac{1}{2}(a_n + jb_n)$
Example
Convert the TFS coefficients of signal $x(t)$ of figure below to EFS coefficients.

From previous TFS:

- $a_0 = 0$
- $a_n = 0$
- $b_n = \begin{cases} 0 & ; n = \text{even} \\ \frac{4}{n\pi} & ; n = \text{odd} \end{cases}$

Example
Convert the TFS coefficients of signal $x(t)$ of figure below to EFS coefficients.

From previous TFS:

- $a_0 = \frac{\pi}{2}$
- $b_n = 0$
- $a_n = \begin{cases} 0 & ; n = \text{even} \\ -\frac{4}{n^2\pi} & ; n = \text{odd} \end{cases}$
Example
Consider the signal $y(t)$ as shown below:

i) Determine the TFS of $y(t)$

ii) From the TFS of $y(t)$, write the EFS of $y(t)$